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Exact Results for the Two-Dimensional, Two-Component Plasma at $\Gamma = 2$ in Doubly Periodic Boundary Conditions

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The two-dimensional, two-component plasma is considered in doubly periodic boundary conditions with the positive and negative charges confined to separate interlacing rectangular lattices. It is shown that at the special coupling $\Gamma = 2$, on a lattice of $2M_1 \times 2M_2$ sites, the grand partition function can be written as a double integral over a product of determinants of dimension $2M_2 \times 2M_2$. On the basis of a conjecture regarding the zero distribution of the grand partition function, the large-M_2 behavior of the determinant is given and the pressure evaluated exactly.

KEY WORDS: Two-component plasma; determinants; Yang–Lee theory.

1. INTRODUCTION

The two-component, two-dimensional (log-potential) Coulomb gas is a model system in many physical theories. The system first became prominent in the work of Kosterlitz and Thouless on topological phase transitions in two dimensions (ref. 1; see also the recent reviews in refs. 2 and 3). As well as the discovery of further topological phase transitions (roughening transition, floating phases, etc.; see, e.g., ref. 4), the Coulomb gas has since been used as the basis for renormalization group theories of two-dimensional phase transitions.[5]

The two-dimensional Coulomb gas in a continuous domain is a two-parameter system: the dimensionless coupling constant

$$\Gamma := q^2/k_B T \tag{1.1}$$

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where q is the magnitude of the charges, and the dimensionless density $\tau\rho$, which is the ratio of the interparticle spacing $1/\rho$ to the "hard-core" diameter of the particles τ. The hard-core or similar regularization of the logarithmic potential is necessary to stop the collapse of positive and negative charge pairs at low temperature.

An alternative to imposing a "hard-core" about each of the charges is to divide the domain into a grid of two sublattices, and allow each species to occupy sites on one or the other of the sublattices. On physical grounds it is expected that the lattice and continuum models will have the same properties in the low-density limit.

One such shared property should be the leading-order singular behavior of the Mayer expansion for $\Gamma > 4$ (the conductor phase) obtained by Zittartz[6] in the continuum for $\zeta \to 0$ as

$$\tau\beta P_{\text{sing}} \sim c(\Gamma) \begin{cases} \zeta^{2/(2d-\Gamma)}, & 1/(2d-\Gamma) \notin \mathbf{Z}^+ \\ \zeta^{2/(2d-\Gamma)} \log \zeta, & 1/(2d-\Gamma) \in \mathbf{Z}^+ \end{cases} \tag{1.2}$$

where ζ denotes the fugacity, the dimension is $d = 2$, and $c(\Gamma)$ is independent of ζ. In the context of the Yang–Lee characterization of a phase transition,[7] the behavior (1.2) implies that in the conductor phase the zeros of the grand partition function must pinch the real axis in the complex fugacity plane at $\zeta = 0$. On the other hand, for $\Gamma \geqslant 4$ the Mayer expansion is convergent,[8] so the neighborhood of $\zeta = 0$ must be zero-free.

A recent study[9] of an analogous two-component, log-potential Coulomb gas on a one-dimensional lattice has shown similar features of the complex zeros to those expected on the two-dimensional lattice. On the one-dimensional lattice, it has been conjectured[10] that all the zeros of the grand partition function lie on the negative real axis in the complex scaled fugacity (ξ) plane for $\Gamma < 2$ and on the unit circle for $\Gamma > 2$. These properties are conjectured to hold true in the finite system, and further, for $\Gamma < 2$, the zero closest to the origin (ξ_1 say) is expected[9] to have the expansion

$$\xi_1 = -\frac{1}{M^{2d-\Gamma}} \left[a_0(\Gamma) + \frac{a_1(\Gamma)}{M} + \frac{a_2(\Gamma)}{M^2} + \cdots \right] \tag{1.3}$$

where M is the order of the grand partition function polynomial and the dimension is $d = 1$. From (1.3) the behavior (1.2) with $d = 1$ is deduced.

Observing the similarities of the behavior (1.2) for $d = 1$ and 2, the question of the behavior of the grand partition function zeros when $d = 2$ immediately arises. This problem is clearly more complex than that in one dimension, as an expansion of the form (1.3) would depend on the precise shape of the finite system in addition to the number of lattice sites. The most obvious starting point toward answering this question is the numerical evaluation of the grand partition function polynomials for small lattice sizes. Experience of our previous study[9] shows that the largest lattice which can be thus treated will have approximately 16 sites. Due to this small size and the anticipated strong shape dependence, it is desirable to have some larger size data available at a particular coupling so as to aid the interpretation of the data in general.

With this aim in mind, in this paper we will make an analytic study of the grand partition function polynomial at the coupling $\Gamma = 2$ in doubly periodic boundary conditions. It will be shown that on a $2M_1 \times 2M_2$ lattice the grand partition function can be written as a double integral over a product of determinants of dimension $2M_2 \times 2M_2$. It is not possible explicitly to evaluate the determinant, but the numerical task of calculating the zeros from this expression now depends only polynomially on M_2 in cost. A numerical study of this formula and the general Γ cases will be undertaken in a subsequent paper.

A spinoff from this calculation is a rederivation of the exact expression for the pressure, in the thermodynamic limit, obtained by Gaudin.[11] Like Gaudin's calculation, our derivation relies on a conjecture so is not rigorous. The conjecture is closely related to the expected behavior of the density of zeros of the grand partition function and thus our main theme.

2. EXACT EVALUATION OF THE GRAND PARTITION FUNCTION AT $\Gamma = 2$

2.1. The Potential in Periodic Boundary Conditions

The solution of the two-dimensional Poisson equation

$$\nabla^2 \phi(x, y) = -2\pi\, \delta(x)\, \delta(y) \tag{2.1}$$

subject to the periodicity condition

$$\phi(x + L, y) = \phi(x, y + W) = \phi(x, y) \tag{2.2}$$

is required.

Formally, this can be achieved by writing

$$\phi(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{2\pi i(mx/L + ny/W)} \tag{2.3}$$

and

$$\delta(x) = \frac{1}{L} \sum_{m=-\infty}^{\infty} e^{2\pi imx/L}, \qquad \delta(y) = \frac{1}{W} \sum_{n=-\infty}^{\infty} e^{2\pi iny/W} \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.1), differentiating under the summation signs, and equating like coefficients of the basis function

$$e^{2\pi i(mx/L+ny/W)}$$

gives

$$a_{m,n}=\frac{1}{2\pi LW}\frac{1}{(m/L)^2+(n/W)^2} \tag{2.5}$$

The coefficient of the constant term $[(m,n)=(0,0)]$ in the Fourier expansion thus diverges. Omitting this term (a step which is often justified by charge neutrality[12]) gives

$$\phi(x,y)=\frac{1}{2\pi LW}\sum_{m=-\infty}^{\infty}\sum_{\substack{n=-\infty\\(m,n)\neq(0,0)}}^{\infty}\frac{e^{2\pi i(mx/L+ny/W)}}{(m/L)^2+(n/W)^2} \tag{2.6}$$

The summation (2.6) is conditionally convergent and the value is sought such that the Poisson equation (2.1) is satisfied. Glasser[13] has summed (2.6) as

$$\phi(x,y)=\pi y^2/LW-\log|\theta_1(\pi(x+iy)/L,e^{-\pi W/L})|+\text{constants} \tag{2.7}$$

where

$$\theta_1(z;q)=-i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+1/2)^2}e^{2i(n+1/2)z} \tag{2.8}$$

A well-known theorem of complex analysis says that the real part of an analytic function is harmonic. Now $\log\theta_1(z;q)$ is analytic in z for $\theta_1\neq 0$ and its real part is $\log|\theta_1(z;q)|$. Hence the term $\pi y^2/LW$ in (2.7) should be omitted for (2.1) to be satisfied. However, the right-hand side of (2.7) without this term is periodic in x, but not in y. The solution satisfying (2.1) obtained from (2.6) is thus discontinuous in the y direction.

The constant term in the potential is chosen so that as L, $W\to\infty$,

$$\phi(x,y)\sim -\tfrac{1}{2}\log(x^2+y^2) \tag{2.9}$$

Since for small $|z|$,

$$\theta_1(z;q)\sim z\theta_1'(0;q) \tag{2.10}$$

then the required potential which satisfies (2.1) and (2.9) is

$$\phi(x,y)=-\log\left|\frac{L\theta_1(\pi(x+iy)/L;e^{-\pi W/L})}{\pi\theta_1'(0,e^{-\pi W/L})}\right| \tag{2.11}$$

Further discussion regarding the solution of (2.1) in doubly periodic boundary conditions can be found in ref. 12.

2.2. Definition of the Model

Consider a rectangle of side lengths L and W. Let the rectangle be divided into a grid of $M_1 \times M_2$ sites, with lattice points at the coordinates $(n_1 L/M_1,\ n_2 W/M_2)$, $n_j = 1, 2,..., M_p$ $(p = 1, 2)$. Introduce a second interlacing lattice with coordinates $((n_1 - \phi_1) L/M_1,\ (n_2 - \phi_2) W/M_2)$ and denote these lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively (see Fig. 1). Allow N $(\leqslant M_1 M_2)$ positive charges to occupy $\mathscr{L}_1$ and N negative charges of the same magnitude to occupy $\mathscr{L}_2$.

Two particles, one of charge e at $\mathbf{x} = (x, y)$ and the other of charge e' at $\mathbf{x}' = (x', y')$ interact via the potential

$$V(\mathbf{x}, \mathbf{x}') = ee'\phi(x - x', y - y') \tag{2.12}$$

where ϕ is given by (2.11). Denote the coordinates of the kth positive charge by $(m_k L/M_1,\ n_k W/M_2)$ and the coordinates of the kth negative charge by $((m_k' - \phi_1) L/M_1,\ (n_k' - \phi_2) W/M_2)$, where $1 \leqslant m_k,\ m_k' \leqslant M_1$ and $1 \leqslant n_k,\ n_k' \leqslant M_2$. Further denote

$$x_k = \pi m_k/M_1 + \pi i W n_k/L M_2 \tag{2.13a}$$

and

$$y_k = \pi(m_k' - \phi_1)/M_1 + \pi i W(n_k' - \phi_2)/L M_2 \tag{2.13b}$$

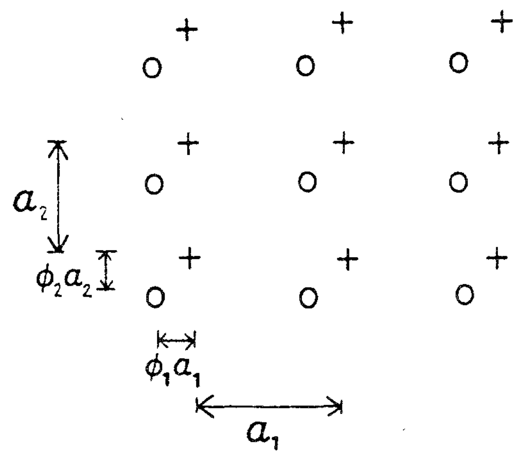

Fig. 1. Geometry of the two interlacing rectangular lattices where $a_1 = L/M_1$ and $a_2 = W/M_2$. The plus signs form $\mathscr{L}_1$, while the circles form $\mathscr{L}_2$.

With the notation (2.13), the Boltzmann factor $W_{N\Gamma}$ for N particles of charge q and N particles of charge $-q$ is

$$W_{N\Gamma}=(|\pi\theta_1'(0;q)/L|)^{N\Gamma}\,|F(x_1,...,x_N;y_1,...,y_N)|^{\Gamma} \tag{2.14}$$

where

$$F(x_1,...,x_N;y_1,...,y_N):=\frac{\prod_{1\leqslant j<k\leqslant N}\theta_1(x_k-x_j;q)\,\theta_1(y_k-y_j;q)}{\prod_{j=1}^{N}\prod_{k=1}^{N}\theta_1(x_j-y_k;q)} \tag{2.15}$$

and

$$q=e^{-\pi W/L} \tag{2.16}$$

The partition function $Z_{N\Gamma}$ and the grand partition function Ξ_Γ are given by

$$Z_{N\Gamma}=\sum_{x\in\{r\}}\sum_{y\in\{s\}}W_{N\Gamma} \tag{2.17}$$

and

$$\Xi_\Gamma=\sum_{N=0}^{M_1M_2}\zeta^{2N}Z_{N\Gamma} \tag{2.18}$$

respectively, where ζ denotes the fugacity and the sum in (2.17) is over the set of combinations of the set of complex numbers $r_{j,k}$ and $s_{j,k}$ taken N at a time, where

$$r_{j,k}=\pi j/M_1+\pi iWk/LM_2 \tag{2.19a}$$

and

$$s_{j,k}=\pi(j-\phi_1)/M_1+\pi iW(k-\phi_2)/LM_2 \tag{2.19b}$$

with $1\leqslant j\leqslant M_1$ and $1\leqslant k\leqslant M_2$.

2.3. Boltzmann Factor as a Determinant

The exact solution at $\Gamma=2$ relies on the following determinant identity, which can be found in ref. 14, Eq. (43), p. 33.

Theorem 2.1. With the notation (2.15),

$$\theta_4\left(\sum_{j=1}^{N}(x_j-y_j)-\alpha;q\right)F(x_1,...,x_N;y_1,...,y_N)$$
$$=\theta_4(\alpha;q)\det\left[\frac{\theta_4(x_j-y_k-\alpha;q)}{\theta_4(\alpha;q)\,\theta_1(x_j-y_k;q)}\right]_{j,k=1,...,N} \tag{2.20}$$

where $\theta_1(z; q)$ is given by (2.8) and

$$\theta_4(z; q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz} \tag{2.21}$$

Proof. (See Appendix A).

Corollary 2.1. We have

$$F(x_1,..., x_N; y_1,..., y_N) = \int_0^1 \theta_4(\pi\alpha; q) \det[f(x_j - y_k; \alpha)]_{j,k=1,...,N} \, d\alpha \tag{2.22}$$

where

$$f(z, \alpha) = \frac{\theta_4(z - \pi\alpha; q)}{\theta_4(\pi\alpha; q)\, \theta_1(z; q)} \tag{2.23}$$

Proof. Simply replace α by $\pi\alpha$, integrate both sides with respect to α, and note that F is independent of α, while

$$\int_0^1 \theta_4(z - \pi\alpha; q)\, d\alpha = 1 \tag{2.24}$$

for any z.

Remark. The determinant identity (2.22), which generalizes the Cauchy double alternant formula, has an analogue (Theorem 4.1 of ref. 15) which generalizes the van der Monde determinant formula.

2.4. Grand Partition Function as a Determinant

Using the key identity (2.22) and the expression for the Boltzmann factor (2.14), we have

$$W_{N2} = (\pi\theta_1'(0; q)/L)^{2N} \int_0^1 \theta_4(\pi\alpha; q) \det[f(x_j - y_k, \alpha)]\, d\alpha \times \int_0^1 \theta_4(\pi\gamma; q) \det[f(\bar{x}_j - \bar{y}_k, \gamma)]\, d\gamma \tag{2.25}$$

where $\bar{z}$ denotes the complex conjugate of z. Next replace the last matrix in (2.25) by its transpose, note from (2.23) that

$$f(-z, \gamma) = -f(z, -\gamma) \tag{2.26}$$

and use the periodicity in γ to conclude that

$$\int_0^1 \theta_4(\pi\gamma; q) \det[f(\bar{x}_j - \bar{y}_k, \gamma)]\, d\gamma$$

$$= (-1)^N \int_0^1 \theta_4(\pi\gamma; q) \det[f(\bar{y}_j - \bar{x}_k, \gamma)]\, d\gamma \tag{2.27}$$

The resulting product of determinants can be written as a block determinant, which gives

$$W_{N2} = (\pi\theta_1'(0, q)/L)^{2N} \times \int_0^1 \int_0^1 \theta_4(\pi\alpha; q)\, \theta_4(\pi\gamma; q) \det\begin{bmatrix} \mathbf{O}_N & f(x_j - y_k, \alpha) \\ f(\bar{y}_j - \bar{x}_k, \gamma) & \mathbf{O}_N \end{bmatrix} d\alpha\, d\gamma \tag{2.28}$$

where $\mathbf{O}_N$ denotes the zero matrix of order N.

If the identity (2.28) is substituted in the expression for the grand partition function (2.17) and (2.18), following Gaudin,[11] we observe that the resulting expression is an expansion in minors of a $2M_1M_2 \times 2M_1M_2$ matrix. We have

$$\Xi_2 = \int_0^1 \int_0^1 \theta_4(\pi\alpha; q)\, \theta_4(\pi\gamma; q) \det(\mathbf{1}_{2M_1M_2} + (\pi\zeta\,|\theta_1'(0, q)|/L)\mathbf{K})\, d\alpha\, d\gamma \tag{2.29}$$

where the matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{bmatrix} \mathbf{O}_{M_1M_2} & f(r_{l_1,l_2} - s_{l_1',l_2'}, \alpha) \\ f(\bar{s}_{l_1,l_2} - \bar{r}_{l_1',l_2'}, \gamma) & \mathbf{O}_{M_1M_2} \end{bmatrix} \tag{2.30}$$

$\mathbf{1}_M$ denotes the $M \times M$ identity matrix and $r_{l,l'}$ and $s_{l,l'}$ are given by (2.19). The integers l_1, l_1', l_2, l_2' lie in the ranges

$$1 \leqslant l_1, l_1' \leqslant M_1 \qquad \text{and} \qquad 1 \leqslant l_2, l_2' \leqslant M_2 \tag{2.31}$$

2.5. Partial Diagonalization of a Block Matrix

Consider the elements of the upper right-hand block of (2.30). From (2.19) the quantity $r_{l_1,l_2} - s_{l_1',l_2'}$ depends only on the differences $l_1 - l_1'$ and $l_2 - l_2'$, so that we can write

$$f(r_{l_1,l_2} - s_{l_1',l_2'}, \alpha) := c^{(0)}_{l_1 - l_1', l_2 - l_2'}(\alpha) \tag{2.32}$$

Further, from (2.23) and the periodicity property (A1.1) of the theta functions given in the Appendix, we have

$$c^{(0)}_{-l_1,l_2}(\alpha) = -c^{(0)}_{M_1-l_1,l_2}(\alpha) \tag{2.33}$$

Similarly, the entries in the lower left-hand block are also dependent on differences, so we can write

$$f(\bar{s}_{l_1,l_2} - \bar{r}_{l'_1,l'_2}, \gamma) := c^{(1)}_{l_1-l'_1,l_2-l'_2}(\gamma) \tag{2.34}$$

These entries have the same "anticyclic" property as those of the upper right-hand block (2.33).

Due to the anticyclic property in the label l_1 it follows that the transformation

$$\begin{bmatrix} U^{-1} & \mathbf{O}_{M_1M_2} \\ \mathbf{O}_{M_1M_2} & U^{-1} \end{bmatrix} \mathbf{K} \begin{bmatrix} U & \mathbf{O}_{M_1M_2} \\ \mathbf{O}_{M_1M_2} & U \end{bmatrix} \tag{2.35}$$

where

$$U = \left[\frac{1}{(M_1M_2)^{1/2}} e^{-2\pi i l'_1(k_1+1/2)/M_1} \delta_{l'_2,k_2} \right] \tag{2.36}$$

(the symbol $\delta_{a,b}$ denotes the Kronecker delta and $0 \leqslant k_2 \leqslant M_2 - 1$) diagonalizes the block matrix $\mathbf{K}$. Since this transformation can be applied to the matrix in (2.29) without changing the value of the determinant, we see from (2.29) and (2.35) that

$$\Xi_2 = \int_0^1 \int_0^1 \theta_4(\pi\alpha; q)\, \theta_4(\pi\gamma; q) \det(\mathbf{1}_{2M_1M_2} + \Lambda)\, d\alpha\, d\gamma \tag{2.37}$$

where

$$\Lambda = \begin{bmatrix} \mathbf{O}_{M_1M_2} & \delta_{k_1,k'_1} S^{(0)}_{k_1,k_2-k'_2}(\alpha) \\ \delta_{k_1,k'_1} S^{(1)}_{k_1,k_2-k'_2}(\gamma) & \mathbf{O}_{M_1M_2} \end{bmatrix} \tag{2.38}$$

and

$$S^{(p)}_{k_1,k_2-k'_2}(\beta) = (\pi\zeta\, |\theta'_1(0; q)|/L) \sum_{l_1=0}^{M_1-1} c^{(p)}_{l_1,k_2-k'_2}(\beta)\, e^{2\pi i l_1(k_1+1/2)/M_1} \tag{2.39}$$

Rearrangement of the rows and columns in the matrix sum $\mathbf{1}_{2M_1M_2} + \Lambda$ shows that the determinant factorizes to give

$$\Xi_2 = \int_0^1 \int_0^1 \theta_4(\pi\alpha; q)\, \theta_4(\pi\gamma; q) \prod_{k_1=0}^{M_1-1} \det \mathbf{A}(k_1)\, d\alpha\, d\gamma \tag{2.40}$$

where

$$\mathbf{A}(k_1)=\mathbf{1}_{2M_2}+\begin{bmatrix} \mathbf{O}_{M_2} & S^{(0)}_{k_1,k_2-k_2'}(\alpha) \\ S^{(1)}_{k_1,k_2-k_2'}(\gamma) & \mathbf{O}_{M_2} \end{bmatrix} \tag{2.41}$$

is a $2M_2\times 2M_2$ block matrix. It therefore remains to evaluate the summations (2.39).

2.6. Fourier Decomposition of a Theta Function Ratio and a Related Summation

To evaluate (2.39), we first write

$$\frac{\theta_4(z-\pi\alpha;q)}{\theta_1(z;q)} \tag{2.42}$$

so that the Fourier components when $z=\pi k/M$, $0\leqslant k\leqslant M-1$, are displayed.

Theorem 2.2. We have

$$\frac{\theta_4(z-\pi\alpha;q)}{\theta_1(z;q)}=\frac{\theta_4(\pi\alpha;q)\,\theta_1'(0;q^M)}{\theta_1(Mz;q^M)\,\theta_1'(0;q)}e^{-Miz} \times\sum_{s=0}^{M-1}e^{(2s+1)iz}\frac{\theta_4(Mz-\pi\alpha-\pi\tau(M/2-s-1/2);q^M)}{\theta_4(\pi\alpha+\pi\tau(M/2-s-1/2);q^M)} \tag{2.43}$$

where

$$q=e^{\pi i\tau} \tag{2.44}$$

Proof. See Appendix A2.

In (2.43) choose

$$z=\frac{\pi}{M_1}(l_1+\phi_1)+\frac{\pi iW}{M_2L}(l_2+\phi_2),\qquad l_2=k_2-k_2'$$
$$q=e^{-\pi W/L},\qquad \tau=iW/L,\qquad M=M_1 \tag{2.45}$$

and substitute the resulting expression in the definitions (2.32) and (2.23) of $c^{(0)}_{l_1,l_2}$. Then the only term dependent on l_1 is the complex exponential $\exp[2\pi i(s+1/2)l_1/M_1]$ [s is the summation variable in (2.43)]. Therefore the summation over l_1 in (2.40) is simply

$$\sum_{l_1=0}^{M_1-1}e^{2\pi il_1(s+k_1+1)/M_1}=M_1\,\delta_{s+k_1+1,M_1} \tag{2.46}$$

and thus

$$\sum_{l_1=0}^{M_1-1} c_{l_1,l_2}^{(0)} \exp \frac{2\pi i l_1(k_1+1/2)}{M_1}$$
$$= M_1 \frac{\theta_1'(0; q^{M_1}) \exp\{2\pi i(M_1/2-k_1-1/2)[\phi_1/M_1 + iW(l_2+\phi_2)/LM_2]\}}{\theta_1'(0;q)\,\theta_1(\pi\phi_1+\pi i W M_1(l_2+\phi_2)/LM_2);\, q^{M_1})}$$
$$\times \frac{\theta_4(\pi\phi_1+\pi i W M_1(l_2+\phi_2)/LM_2 - \pi\alpha + \pi\tau(M_1/2-k_1-1/2);\, q^{M_1})}{\theta_4(\pi\alpha - \pi\tau(M_1/2-k_1-1/2);\, q^{M_1})}$$
$$:= f\left(\phi_1, \phi_2, \frac{k_1+1/2}{M_1}, l_2, \alpha\right) \tag{2.47}$$

Similar use of (2.43) shows that

$$\sum_{l_1=0}^{M_1-1} c_{l_1,l_2}^{(1)} e^{2\pi i l_1(k_1+1/2)/M_1} = f(-\phi_1, \phi_2, (k_1+1/2)/M_1, -l_2, \gamma) \tag{2.48}$$

where f is defined in (2.47).

2.7. The Scaled Fugacity

So that the particle–hole symmetry of the lattice gas is most clearly displayed, it is necessary to scale the fugacity ζ so that the coefficient of $\zeta^{2M_1M_2}$ is unity. This can be achieved by calculating the energy of the configuration in which all lattice sites are filled and scaling by the corresponding Boltzmann factor.

For a potential periodic in both directions the energy when all sites are filled is the same at each site. From Section 2.1 we know that the potential

$$\phi(x, y) = \pi y^2/LW - \log|\theta_1(\pi(x+iy)/L;\, e^{-\pi W/L})| \tag{2.49}$$

is periodic in both x and y. After introducing the notation

$$P(e^{-\pi r}) = \prod_{\substack{l_1=0 \\ (l_1,l_2)\neq(0,0)}}^{M_1-1} \prod_{l_2=0}^{M_2-1} \theta_1(\pi(l_1/M_1 + i l_2 r/M_2);\, e^{-\pi r}) \tag{2.50}$$

and

$$Q(\phi_1, \phi_2; e^{-\pi r}) = \prod_{l_1=0}^{M_1-1} \prod_{l_2=0}^{M_2-1} \theta_1(\pi[(l_1+\phi_1)/M_1 + i(l_2+\phi_2)r/M_2];\, e^{-\pi r}) \tag{2.51}$$

we can write the energy $E^{(1)}(p)$ at a site in row p ($p = 0, 1, \ldots, M_2 - 1$) of sublattice $\mathscr{L}_1$ with this potential when all lattice sites are filled as

$$E^{(1)}(p) = E_1^{(1)}(p) + E_2^{(1)}(p) \tag{2.52}$$

where

$$E_1^{(1)}(p) = \frac{-q^2 \pi W M_1}{L(M_2)^2} \sum_{l_2=0}^{M_2-1} [(l_2 - p)^2 - (l_2 + \phi_2 - p)^2] \tag{2.53}$$

and

$$E_2^{(1)}(p) = -q^2 \log |P(e^{-\pi W/L})/Q(\phi_1, \phi_2; e^{-\pi W/L})| \tag{2.54}$$

Since $E^{(1)}(p)$ is independent of p, we have

$$E^{(1)}(p) = E_1^{(1)}(0) + E_2^{(1)}(0) \tag{2.55}$$

and thus

$$E_2^{(1)}(p) = E_1^{(1)}(0) + E_2^{(1)}(0) - E_1^{(1)}(p) \tag{2.56}$$

The quantity $E_2^{(1)}(p)$ is the energy at a site in row p of the sublattice $\mathscr{L}_1$ with all sites filled and the potential (2.49) without the term $\pi y^2/LW$, which is the Coulomb potential of Section 2.1. Since this potential is not periodic in the y direction, $E_2^{(1)}(p)$ will depend on p.

To calculate $E_2^{(1)}(p)$, it follows from the product expansion of θ_1.

$$P(q) = i^{M_2 - 1} M_1 q^{-M_1 M_2/4 + M_1/2 - 1/4} \prod_{n=1}^{\infty} (1 - q^{2nM_1/M_2})^2 (1 - q^{2n})^{M_1 M_2 - 3} \tag{2.57}$$

and

$$\begin{aligned} Q(\phi_1, \phi_2; q) = & -i^{M_2+1} q^{-M_1 M_2/4 - M_1/4M_2 + M_1/2} e^{\pi r \phi_2 M_1(1 - 1/M_2) - \pi i \phi_1 (M_2 - 1)} \\ & \times \theta_1(\pi\phi_1 + \pi i r M_1 \phi_2/M_2; q^{M_1/M_2}) \\ & \times \prod_{n=1}^{\infty} (1 - q^{2n})^{M_1 M_2}/(1 - q^{2nM_1/M_2}) \end{aligned} \tag{2.58}$$

Hence from (2.56), (2.54), and (2.53) we have

$$E_2^{(1)}(p) = -q^2 \log \left| \frac{M_1 q^{\phi_2 M_1 - \phi_2 M_1 (2p+1)/M_2} \theta_1'(0; q^{M_1/M_2})}{\theta_1'(0; q)\, \theta_1(\pi\phi_1 + \pi i W M_1 \phi_2/L M_2; q^{M_1/M_2})} \right| \tag{2.59}$$

where

$$q = e^{-\pi W/L} \tag{2.60}$$

A similar calculation shows that the analogous energy $E_2^{(2)}(p)$ on a site of row p of sublattice $\mathscr{L}_2$ is given by (2.59) with the replacements $\phi_1 \mapsto -\phi_1$ and $\phi_2 \mapsto -\phi_2$, so that the combination $E_2^{(1)}(p)+E_2^{(2)}(p)$ is independent of p. The total energy E of the completely filled lattice is therefore

$$\begin{aligned} E &= \tfrac{1}{2} M_1 M_2 (E_2^{(1)}(p)+E_2^{(2)}(p)) \\ &= -M_1 M_2 q^2 \log \left| \frac{M_1 \theta_1'(0; q^{M_1/M_2})}{\theta_1'(0, q)\, \theta_1(\pi\phi_1 + \pi i W M_1 \phi_2/L M_2; q^{M_1/M_2})} \right| \end{aligned} \tag{2.61}$$

If we replace the fugacity ζ by the scaled fugacity ξ according to the formula

$$\xi = \zeta^2 |\pi \theta_1'(0; q)/L|^{\Gamma} e^{-\Gamma E/(M_1 M_2 q^2)} \tag{2.62}$$

with E given by (2.61), the coefficient of the highest power of ξ in the grand partition function (2.18) is unity, as required.

2.8. Final Expression for Ξ_2

The formula (2.40), with the scaled fugacity as specified by (2.62) and (2.61) and the summation evaluations (2.47) and (2.48) is our final expression for Ξ_2. Thus we have

$$\Xi_2 = \int_0^1 \int_0^1 \theta_4(\pi\alpha; q)\, \theta_4(\pi\gamma; q) \prod_{k_1=0}^{M_1-1} \det \mathbf{A}(k_1)\, d\alpha\, d\gamma$$

where

$$\mathbf{A}(k_1) = \mathbf{1}_{2M_2} + \begin{bmatrix} \mathbf{O}_{M_2} & cf(\phi_1, \phi_2, (k_1+1/2)/M_1, k_2 - k_2', \alpha) \\ cf(-\phi_1, \phi_2, (k_1+1/2)/M_1, k_2' - k_2, \gamma) & \mathbf{O}_{M_2} \end{bmatrix} \tag{2.63}$$

and

$$c = \xi^{1/2} \left| \frac{\theta_1(\pi\phi_1 + \pi i W M_1 \phi_2/L M_2; q^{M_1/M_2})\, \theta_1'(0; q)}{M_1 \theta_1'(0; q^{M_1/M_2})} \right| \tag{2.64}$$

The function f is specified by (2.47), and the rows of the upper right-hand and lower left-hand blocks in (2.63) are labeled by k_2 $(0 \leqslant k_2 \leqslant M_2 - 1)$, while the columns are labeled by k_2' $(0 \leqslant k_2' \leqslant M_2 - 1)$.

3. THE THERMODYNAMIC LIMIT

3.1. The Infinite-Strip Limit

Two different limits give an infinite strip of lattice points,

$$M_1, L \to \infty, \qquad L/M_1 \to a_1 \tag{3.1a}$$

and

$$M_2,\ W \to \infty, \qquad W/M_2 \to a_2 \tag{3.1b}$$

From Section 2.1 we know that the potential is periodic in the direction of the strip for (3.1a), but not for (3.1b). In the latter case the potential $\phi(y)$ in the direction of the strip is equal to the periodic function of case (3.1a) plus the quadratic $\pi y^2/LW$. It is therefore to be expected that the free energy in each case will be different.

We will first consider the limit (3.1a). The product in (2.40) is then essentially a Riemann sum approximation to an integral. Also, from the conjugate modulus transformation

$$\theta_4(z; e^{\pi i\tau}) = (-i\tau)^{-1/2}\, e^{-iz^2/\pi\tau}\theta_2(z/\tau; e^{-\pi i/\tau}) \tag{3.2}$$

it follows that for $L \to \infty$,

$$\theta_4(\pi\alpha; e^{-\pi W/L}) \sim \left(\frac{L}{\pi W}\right)^{1/2} e^{-\pi L(\alpha - 1/2)^2/W} \tag{3.3}$$

Hence

$$\begin{aligned} \Xi_2 \sim \left(\frac{L}{\pi W}\right) \int_0^1 d\alpha \int_0^1 d\gamma \exp\Bigg\{ -\frac{\pi L}{W}\left[\left(\alpha - \frac{1}{2}\right)^2 + \left(\gamma - \frac{1}{2}\right)^2\right] \\ + M_1 \int_0^1 \log[\det \mathbf{A}(\alpha, \gamma, x)]\, dx \Bigg\} \end{aligned} \tag{3.4}$$

where

$$\mathbf{A}(\alpha, \gamma, x) = \mathbf{1}_{2M_2} + \begin{bmatrix} \mathbf{O}_{M_2} & g(\phi_1, \phi_2, x, k_2 - k_2', \alpha) \\ g(-\phi_1, \phi_2, x, k_2' - k_2, \gamma) & \mathbf{O}_{M_2} \end{bmatrix} \tag{3.5}$$

and

$$\begin{aligned} &g(\phi_1, \phi_2, x, l_2, \alpha) \\ &\quad = \xi^{1/2} e^{2\pi i(1/2 - x)[\phi_1 + ia_2(l_2 + \phi_2)/a_1]} \\ &\qquad \times \frac{\theta_1'(0; q^{M_1})}{\theta_1'(0; q^{M_1/M_2})} \frac{|\theta_1(\pi\phi_1 + \pi i W M_1 \phi_2/LM_2; q^{M_1/M_2})|}{\theta_1(\pi\phi_1 + \pi i W M_1(l_2 + \phi_2)/LM_2; q^{M_1})} \\ &\qquad \times \frac{\theta_4(\pi\phi_1 + \pi i a_2(l_2 + \phi_2)/a_1 - \pi\alpha + \pi i W(1/2 - x)/a_1; e^{-\pi W/a_1})}{\theta_4(\pi\alpha - \pi i W(1/2 - x)/a_1; e^{-\pi W/a_1})} \end{aligned} \tag{3.6}$$

The pressure P_W in an infinite strip of width W at $\Gamma=2$, defined as

$$\beta P_W = \lim_{\substack{M_1, L \to \infty \\ L/M_1 \to a_1}} \frac{1}{L} \log \Xi_2 \tag{3.7}$$

is therefore given by the formula

$$\beta P_W = \mathrm{Re}\left\{\frac{\pi}{W}\left[\left(\alpha_0 - \frac{1}{2}\right)^2 + \left(\gamma_0 - \frac{1}{2}\right)^2\right] - \frac{1}{a_1}\int_0^1 \log[\det \mathbf{A}(\alpha_0, \gamma_0, x)]\,dx\right\} \tag{3.8}$$

where α_0 and γ_0 maximize the integral (3.4).

To obtain the strip free energy in the limit (3.16), it would be necessary to provide the large-M_2 behavior of the $2M_2 \times 2M_2$ matrix $\mathbf{A}(k_1)$ as given by (2.63). It follows from the periodicity property (A1.2) of the theta functions that the elements f as given by (2.47) of the block matrix in (2.63) have the periodicity property

$$f(\phi_1, \phi_2, l_2 + M_2, \alpha) = e^{2\pi i \alpha} f(\phi_1, \phi_2, l_2, \alpha) \tag{3.9}$$

Since in general $\alpha \neq \gamma$, the matrix $\mathbf{A}(k_1)$ has no simple cyclic structure. We have been unable to obtain the asymptotic behavior of this matrix.

3.2. The Bulk Pressure

To obtain the bulk pressure from the strip pressure (3.8), the values of α_0 and γ_0 for large strip width W are required. To deduce this behavior, we use the physical argument that the strip system, having an effective one-dimensional Coulomb potential, should be in a dipole phase. As such, $P_W(\xi)$, in the complex ξ plane, must be free of singularities in a neighborhood of $\xi = 0$. However, as the strip width is increased, the effective one-dimensional potential gives way to the two-dimensional logarithmic Coulomb potential. At $\Gamma = 2$, in the bulk, the system is a conductor and the singular behavior (1.2) implies that the singularities pinch the point $\xi = 0$.

From (3.8) the singularities of $f_W(\xi)$ occur at the zeros of $\mathbf{A}(\alpha_0, \gamma_0, x)$. From (3.5) and (3.6), for $\mathbf{A}(\alpha_0, \gamma_0, x)$ to approach zero as $|\xi| \to 0$, the matrix elements as defined by (3.6) must tend to infinity. This occurs when the denominators

$$\theta_4(\pi\alpha_0 - \pi i W(1/2 - x); e^{-\pi W/a_1}), \qquad \theta_4(\pi\gamma_0 - \pi i W(1/2 - x); e^{-\pi W/a_1}) \tag{3.10}$$

approach zero. The function $\theta_4(\pi z; e^{\pi i\tau})$ vanishes at $z=(\pi\tau/2) \bmod \pi, \pi\tau$, and thus (3.10) can only vanish if $\alpha_0, \gamma_0 = 0$ or 1 [the points 0 and 1 are equivalent, since the integrand in (3.4) has period 1 in both α and γ]. Thus, we expect α_0 and γ_0 to tend to zero in the limit $W \to \infty$.

With $\alpha_0=\gamma_0=0$, from (3.5) and (3.9), the matrix $\mathbf{A}(\alpha_0, \gamma_0, x)$ is block cyclic and thus diagonalized by the transformation

$$\begin{bmatrix} \mathbf{V}^{-1} & \mathbf{O}_{M_2} \\ \mathbf{O}_{M_2} & \mathbf{V}^{-1} \end{bmatrix} \mathbf{A}(0,0,x) \begin{bmatrix} \mathbf{V} & \mathbf{O}_{M_2} \\ \mathbf{O}_{M_2} & \mathbf{V} \end{bmatrix} \tag{3.11}$$

where

$$\mathbf{V} = \left[\frac{1}{\sqrt{M_2}} e^{-2\pi i k_2' p/M_2} \right]_{k_2', p=0,1,\ldots,M_2-1} \tag{3.12}$$

The resulting determinant is simply evaluated to show that for large W and M_2,

$$\beta P_W \sim \frac{1}{a_1} \int_0^1 \log \prod_{p=0}^{M_2-1} [1 - h_+(\phi_1, \phi_2, x, p)\, h_-(-\phi_1, \phi_2, x, p)]\, dx \tag{3.13}$$

where

$$h_\pm(\phi_1, \phi_2, x, p) = \sum_{l_2=0}^{M_2-1} g(\phi_1, \phi_2, x, l_2, 0) e^{\pm 2\pi i l_2 p/M_2} \tag{3.14}$$

To evaluate the summations (3.14), we first apply the conjugate modulus transformations (3.2),

$$\theta_1(z; e^{\pi i\tau}) = -i(-i\tau)^{-1/2} e^{-iz^2/\pi\tau} \theta_1(z/\tau; e^{-\pi i/\tau}) \tag{3.15a}$$

and

$$\theta_1'(0; e^{-\pi\varepsilon}) = (1/\varepsilon)^{3/2} \theta_1'(0; e^{-\pi/\varepsilon}) \tag{3.15b}$$

together with the formula

$$\theta_2(z; q) = q^{1/4} e^{iz} \theta_4(z + \pi/2 + \pi\tau/2; q) \tag{3.16}$$

to rewrite g, as defined by (3.6), as

$$\begin{aligned} g(\phi_1, \phi_2, x, l_2, 0) &= \xi^{1/2} \frac{i}{M_2} e^{-\pi a_1 \phi_1^2/a_2 + \pi a_2 \phi_2^2/a_1} \frac{\theta_1'(0; e^{-\pi a_1/W})}{\theta_1'(0; e^{-\pi a_1/a_2})} \\ &\quad \times e^{\pi a_1 \phi_1/W + \pi i(l_2+\phi_2)/M_2} |\theta_1(\pi\phi_2 + \pi i a_1 \phi_1/a_2; e^{-\pi a_1/a_2})| \\ &\quad \times \frac{\theta_4(z' - \pi\alpha'; q')}{\theta_4(\pi\alpha'; q')\, \theta_1(z'; q')} \end{aligned} \tag{3.17}$$

where

$$z' = \frac{\pi}{M_2}(l_2 + \phi_2) - \frac{\pi i a_1}{W}\phi_1$$
$$\alpha' = x - ia_1/W, \qquad q' = e^{-\pi a_1/W} \tag{3.18}$$

The term involving the modulus in (3.17) is a factor in both h_+ and h_-. We can remove the modulus if we replace the term by its complex conjugate in h_-. Use of (2.43), together with the interrelationship

$$\theta_1(z; e^{\pi i\tau}) = -iq^{1/4}e^{iz}\theta_4(z + \pi\tau/2; e^{\pi i\tau}) \tag{3.19}$$

then shows that, to leading order in M_2,

$$h_+(\phi_1, \phi_2, x, p) \sim \xi^{1/2}H(\phi_1, \phi_2, x, p/M_2) \tag{3.20a}$$

$$h_-(\phi_1, \phi_2, x, p) \sim -\xi^{1/2}\bar{H}(\phi_1, \phi_2, x, p/M_2) \tag{3.20b}$$

where

$$H(\phi_1, \phi_2, x, y) = \exp\left[-\frac{\pi a_1 \phi_1^2}{a_2} + \frac{\pi a_2 \phi_2^2}{a_1} - 2\pi y\left(\frac{a_1\phi_1}{a_2} + i\phi_2\right)\right] \times \frac{\theta_1(\pi(x - \phi_2) + \pi i a_1(y + \phi_1)/a_2; \exp(-\pi a_1/a_2))}{\theta_1(\pi x + \pi i a_1 y/a_2; \exp(-\pi a_1/a_2))} \tag{3.21}$$

and we recall that $a_1 := L/M_1$ and $a_2 := W/M_2$.

Hence, after substituting (3.20) in (3.13) and observing that the resulting expression is a Riemann approximation to a definite integral, we see that the bulk pressure P_B is given by

$$\beta P_B = \frac{1}{a_1 a_2}\int_0^1 \int_0^1 \log[1 + \xi |H(\phi_1, \phi_2, x, y)|^2]\, dx\, dy \tag{3.22}$$

In the special case $\theta_1 = \theta_2 = 1/2$ this formula is equivalent to that given by Gaudin [ref. 11, Eq. (53); see also ref. 16].

APPENDIX

A1. Proof of Theorem 2.1

First we note that the lhs of (2.20) is antisymmetric in each of the x_k and in each of the y_k. The rhs also has this property, since interchanging, say, x_l with $x_{l'}$ is equivalent to interchanging two rows of the determinant, which changes the sign. Interchanging x_k with y_k for each $k=1,\dots,N$ on the lhs is equivalent to replacing α by $-\alpha$ and multiplying by $(-1)^N$. This property clearly holds true on the rhs. It thus suffices to prove that both sides of the equation are the same function of x_1 and α.

To do this, we will use Liouville's theorem. Since

$$\theta_1(z+\pi;q) = -\theta_1(z;q), \qquad \theta_4(z+\pi;q) = \theta_4(z;q) \tag{A1.1}$$

under the translation $x_1 \mapsto x_1 + \pi$ both the lhs and rhs are unchanged apart from a factor of -1 on both sides. Next consider the periodicity of both sides under the mapping $x_1 \mapsto x_1 + \pi\tau$, where $q = e^{\pi i\tau}$ and $\mathrm{Im}(\tau) > 0$. Since

$$\begin{aligned}\theta_1(x+\pi\tau;q) &= -q^{-1}e^{-2ix}\theta_1(x;q)\\ \theta_4(x+\pi\tau;q) &= -q^{-1}e^{-2ix}\theta_4(x;q)\end{aligned} \tag{A1.2}$$

both the lhs and rhs remain unchanged, apart from a factor of $e^{2i\alpha}$ on both sides, under this translation.

Finally, consider the ratio rhs/lhs. From the above results, this is a doubly periodic function of x_1 with periods π and $\pi\tau$. Furthermore, since the rhs vanishes at the zeros of the lhs (which are all simple), we have that rhs/lhs is a doubly periodic entire function of x_1 and thus, by Liouville's theorem, equal to a function $g(\alpha)$ independent of x_1 (and thus $x_2,\dots,x_N$ and the y_k).

To see that $g(\alpha)=1$, consider the limit $y_j \to x_j$ for each $j=1,\dots,N$. In the determinant the diagonal term dominates, so that the rhs behaves as

$$\theta_4(\alpha;q)\prod_{j=1}^{N}\frac{1}{\theta_1(x_j-y_j;q)} \tag{A1.3}$$

Inspection of the lhs shows this to be the leading-order term, so $g(\alpha)=1$, as claimed. The identity (2.20) is thus proved.

A2. Proof of Theorem 2.2

We seek the Fourier decomposition of the function

$$f(z) = \frac{\theta_4(z-\pi\alpha;q)}{\theta_1(z-i\mu;q)}, \qquad q = e^{\pi i\tau}, \quad \mu > 0 \tag{A2.1}$$

in the limit $\mu \to 0^+$. The function is antiperiodic (antiperiod π) and continuous for all real z, so therefore has the Fourier decomposition

$$f(z) = \frac{1}{\pi}\sum_{m=-\infty}^{\infty} a_m(\alpha,\mu,q)e^{(2m+1)iz} \tag{A2.2}$$

where

$$a_m(\alpha, \mu; q) = \int_{-\pi/2}^{\pi/2} \frac{\theta_4(z - \pi\alpha; q)}{\theta_1(z - i\mu; q)} e^{-(2m+1)iz} \, dz \tag{A2.3}$$

Consider

$$I := \int_C \frac{\theta_4(z - \pi\alpha; q)}{\theta_1(z - i\mu; q)} e^{-(2m+1)iz} \, dz \tag{A2.4}$$

where C is the rectangle with vertices at $\pi/2$, $-\pi/2$, $\pi/2 + \pi\tau$, $-\pi/2 + \pi\tau$ traversed anticlockwise. By the properties (A1.2) and the formula

$$e^{-(2m+1)\pi i\tau} = q^{-(2m+1)} \tag{A2.5}$$

parametrization of the paths parallel to the real axis gives

$$I = a_m(\alpha, \mu; q)(1 - q^{(-2m+1)} e^{2\pi i\alpha + 2\mu}) \tag{A2.6}$$

(Note: The contribution to I from the two sides parallel to the imaginary axis cancel by the periodicity of the integrand.)

On the other hand, the contour C encloses a single simple pole at $z = i\mu$, so the residue theorem gives

$$I = 2\pi i \frac{\theta_4(i\mu - \pi\alpha; q)}{\theta_1'(0; q)} e^{(2m+1)\mu} \tag{A2.7}$$

Equating (A2.6) and (A2.7) thus gives the value of $a_m(\mu, \alpha; q)$. Substituting this value in (A2.2) gives

$$f(z) = -2i \frac{\theta_4(i\mu - \pi\alpha; q)}{\theta_1'(0; q)} \sum_{m=-\infty}^{\infty} \frac{q^{(2m+1)} e^{(2m+1)\mu - 2\pi i\alpha - 2\mu} e^{(2m+1)iz}}{1 - q^{(2m+1)} e^{-2\pi i\alpha - 2\mu}} \tag{A2.8}$$

If we write

$$m = Mn + s, \qquad 0 \leqslant s \leqslant M - 1, \qquad n = 0, \pm 1, \ldots \tag{A2.9}$$

and replace the sum over m by a sum over n and s, then (A2.8) becomes

$$f(z) = -2i \frac{\theta_4(i\mu - \pi\alpha; q)}{\theta_1'(0; q)} \sum_{s=0}^{M-1} g(s) e^{(2s+1)iz} \tag{A2.10}$$

where

$$g(s) = \sum_{n=-\infty}^{\infty} \frac{q^{(2Mn+2s+1)} e^{(2Mn+s+1)\mu - 2\pi i\alpha - 2\mu} e^{2Mniz}}{1 - q^{(2Mn+2s+1)} e^{-2\pi i\alpha - 2\mu}} \tag{A2.11}$$

Comparison of (A2.11) with the sum over m in (A2.8) shows that the summands are of the same functional form. Hence $g(s)$ can be summed as

$$g(s)=\frac{1}{2}\, ie^{\mu(s-M+1)} \frac{\theta_1'(0; q^M)e^{-iMz}}{\theta_4(i\mu M-[\pi\alpha+\mu(1-M)+\pi\tau(M-1)/2-\pi\tau s]; q^M)} \times \frac{\theta_4(Mz-[\pi\alpha+\mu(1-M)+\pi\tau(M-1)/2-\pi\tau s]; q^M)}{\theta_1(Mz-i\mu M; q^M)} \qquad \text{(A2.12)}$$

Substituting (A2.12) in (A2.10) and taking the limit $\mu\to 0^+$ gives the required result. ▌

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